**Suggested solutions to the Contract Theory exam on Jan. 6, 2012**
**VERSION: 30 January 2012**

**Question 1 (adverse selection)**

**The following is a model of an insurance market with adverse selection. It builds on the standard adverse selection model that we studied in the course.**

**The principal (P) is a monopoly insurance company and the agent (A) is a car owner who may want to take a car insurance. Depending on how skillful A is as a driver, she may or may not have an accident. The probability of having an accident depends on A's type. A skillful (and therefore a low-demand) driver has an accident with probability $\underline{\theta}$, and a less skillful (and therefore a high-demand) driver has an accident with probability $\overline{\theta}$. Assume that $0 < \underline{\theta} < \overline{\theta} < 1$.**

**A's disutility of having an accident, measured in monetary terms as a deduction from her income, is denoted $d > 0$, and A's monetary income is denoted $w > d$. Moreover, A's payment to P in case there is no accident is denoted $p$; and the net compensation A receives from P in case there indeed is an accident is denoted $a$. A is risk averse and her utility function is denoted $u$ (where $u' > 0$ and $u'' < 0$). Therefore, A's utility if taking the insurance is**

$$\begin{cases} u(w-d+a) & \text{if having an accident} \\ u(w-p) & \text{if not having an accident.} \end{cases}$$

**P is risk neutral and wants to maximize its expected profits. It does not know the type of A, but assigns the probability $v \in (0,1)$ to the event that $\theta = \underline{\theta}$.**

**P offers a menu of two distinct contracts to A. As in the course, the contract variables are indicated either with "upper-bars" or "lower-bars", depending on which type the contract is aimed at. The contract variables are $p$ and $a$. However, to solve the problem it is more convenient to think of P as choosing the utility levels directly, instead of the contract variables. Thus introduce the following notation:**

$$\overline{u}_N \equiv u(w-\overline{p}), \quad \overline{u}_A \equiv u(w-d+\overline{a}), \quad \underline{u}_N \equiv u\left(w-\underline{p}\right), \quad \underline{u}_A \equiv u(w-d+\underline{a}).$$

**Also let $h$ be the inverse of $u$ (hence $h' > 0$ and $h'' > 0$). We can now rewrite the problem as follows. Given that P is risk neutral and wants to maximize its expected profit, P's objective function can be written as**

$$\begin{aligned} V &= v\left[(1-\underline{\theta})\underline{p} - \underline{\theta}\underline{a}\right] + (1-v)\left[\left(1-\overline{\theta}\right)\overline{p} - \overline{\theta}\overline{a}\right] \\ &= v\left[w - \underline{\theta}d - (1-\underline{\theta})h(\underline{u}_N) - \underline{\theta}h(\underline{u}_A)\right] \\ &\quad + (1-v)\left[w - \overline{\theta}d - \left(1-\overline{\theta}\right)h(\overline{u}_N) - \overline{\theta}h(\overline{u}_A)\right]. \end{aligned}$$

**P wants to maximize $V$ w.r.t. $(\underline{u}_N, \underline{u}_A, \overline{u}_N, \overline{u}_A)$, subject to the following four constraints:**

$$\left(1-\overline{\theta}\right)\overline{u}_N + \overline{\theta}\overline{u}_A \geq \overline{U}^*, \qquad \text{(IR-high)}$$

$$(1-\underline{\theta})\,\underline{u}_N + \underline{\theta}\underline{u}_A \geq \underline{U}^*, \tag{IR-low}$$

$$\left(1-\overline{\theta}\right)\overline{u}_N + \overline{\theta}\overline{u}_A \geq \left(1-\overline{\theta}\right)\underline{u}_N + \overline{\theta}\underline{u}_A, \tag{IC-high}$$

$$(1-\underline{\theta})\,\underline{u}_N + \underline{\theta}\underline{u}_A \geq (1-\underline{\theta})\,\overline{u}_N + \underline{\theta}\overline{u}_A, \tag{IC-low}$$

**where**

$$\overline{U}^* \equiv \left(1-\overline{\theta}\right)u(w) + \overline{\theta}u(w-d)\,, \quad \underline{U}^* \equiv (1-\underline{\theta})\,u(w) + \underline{\theta}u(w-d)$$

**are the two types' outside options.**

**a) At the first-best optimum (i.e., the optimum when A's type is observable), both types are offered a contract with full insurance (so that $\overline{u}_N = \overline{u}_A$ and $\underline{u}_N = \underline{u}_A$). Explain, in words, the economic logic behind this result.**

- Two crucial assumptions that lead to this result are that (i) A is risk averse and (ii) P is risk neutral. The objective of P is to maximize its (expected) payoff. Under first best, the only constraints are the individual rationality constraints. Therefore, it is in the interest of P to choose A's level of insurance (for any given price A must pay for this insurance) in a way that makes A's payoff as large as possible, at least as long as this can be done at no cost for P. For if A's payoff from the insurance is higher, then P can charge more for the insurance without making A prefer his outside option. Given that A is risk averse and P is risk neutral, providing A with more insurance leads to a higher payoff for A at no cost for P. Hence the first-best optimum involves P providing full insurance to A and then choosing the effective price for this insurance so high that each type of A is indifferent between the outside option and the insurance contract.
  - The reason why the logic above does not apply under second best is that then P has a smaller number of instruments available: P cannot observe A's type, which means that the level of A's insurance must also be such that A voluntarily chooses the right contract.

**b) Show that the constraints (IC-high) and (IC-low) jointly imply that $\underline{u}_N - \underline{u}_A \geq \overline{u}_N - \overline{u}_A$.**

- Add up the ICs:

  $$\left(1-\overline{\theta}\right)\overline{u}_N+\overline{\theta}\overline{u}_A+(1-\underline{\theta})\,\underline{u}_N+\underline{\theta}\underline{u}_A \geq \left(1-\overline{\theta}\right)\underline{u}_N+\overline{\theta}\underline{u}_A+(1-\underline{\theta})\,\overline{u}_N+\underline{\theta}\overline{u}_A.$$

  Re-arranging and noticing that some terms cancel out, we obtain

  $$-\left(\overline{\theta}-\underline{\theta}\right)\overline{u}_N + \left(\overline{\theta}-\underline{\theta}\right)\overline{u}_A + \left(\overline{\theta}-\underline{\theta}\right)\underline{u}_N - \left(\overline{\theta}-\underline{\theta}\right)\underline{u}_A \geq 0.$$

  Since $\overline{\theta} > \underline{\theta}$, the inequality simplifies to

  $$-\overline{u}_N + \overline{u}_A + \underline{u}_N - \underline{u}_A \geq 0$$

  or

  $$\underline{u}_N - \underline{u}_A \geq \overline{u}_N - \overline{u}_A,$$

  which we were asked to show.

c) **Assume that the constraints (IR-high) and (IC-low) are lax at the second-best optimum (so that they can be disregarded). Show that, at the second-best optimum, the high type is fully insured ($\overline{u}_N = \overline{u}_A$) whereas the low-type is underinsured ($\underline{u}_N > \underline{u}_A$).**

- The Lagrangian:

$$\begin{aligned}\mathcal{L} &= \upsilon \left[w - \underline{\theta} d - (1-\underline{\theta}) h(\underline{u}_N) - \underline{\theta} h(\underline{u}_A)\right] + (1-\upsilon)\left[w - \overline{\theta} d - \left(1-\overline{\theta}\right) h(\overline{u}_N) - \overline{\theta} h(\overline{u}_A)\right] \\ &\quad + \lambda \left[(1-\underline{\theta}) \underline{u}_N + \underline{\theta} \underline{u}_A - \underline{U}^*\right] + \mu \left[\left(1-\overline{\theta}\right) \overline{u}_N + \overline{\theta} \overline{u}_A - \left(1-\overline{\theta}\right) \underline{u}_N - \overline{\theta} \underline{u}_A\right],\end{aligned}$$

  where $\lambda$ is the shadow price associated with IR-low and $\mu$ is the shadow price associated with IC-high.

- FOC w.r.t. $\overline{u}_N$:

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_N} = -(1-\upsilon)\left(1-\overline{\theta}\right) h'(\overline{u}_N) + \mu\left(1-\overline{\theta}\right) = 0$$

  or

$$\boxed{(1-\upsilon) h'(\overline{u}_N) = \mu.} \tag{1}$$

  - This implies that $\boxed{\mu > 0}$; i.e., IC-high binds at the optimum.

- FOC w.r.t. $\underline{u}_N$:

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_N} = -\upsilon (1-\underline{\theta}) h'(\underline{u}_N) + \lambda (1-\underline{\theta}) - \mu\left(1-\overline{\theta}\right) = 0$$

  or

$$\boxed{\upsilon (1-\underline{\theta}) h'(\underline{u}_N) = \lambda (1-\underline{\theta}) - \mu\left(1-\overline{\theta}\right).} \tag{2}$$

  - This implies that $\boxed{\lambda > 0}$ (spell out the arguments!); i.e., IR-low binds at the optimum.

- FOC w.r.t. $\overline{u}_A$:

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_A} = -(1-\upsilon)\overline{\theta} h'(\overline{u}_A) + \mu\overline{\theta} = 0$$

  or

$$\boxed{(1-\upsilon) h'(\overline{u}_A) = \mu.} \tag{3}$$

- FOC w.r.t. $\underline{u}_A$:

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_A} = -\upsilon\underline{\theta} h'(\underline{u}_A) + \lambda\underline{\theta} - \mu\overline{\theta} = 0$$

  or

$$\boxed{\upsilon\underline{\theta} h'(\underline{u}_A) = \lambda\underline{\theta} - \mu\overline{\theta}.} \tag{4}$$

- Combining (1) and (3) immediately yields (here we use $h' > 0$)

$$\overline{u}_N = \overline{u}_A \equiv \overline{u}.$$

  - That is, full insurance for the high type, which was one of the results we were asked to show.

- Multiply (2) by $\underline{\theta}$:

$$\upsilon\underline{\theta}\left(1-\underline{\theta}\right)h'\left(\underline{u}_N\right)=\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\underline{\theta}\left(1-\overline{\theta}\right).$$

- Multiply (4) by $(1-\underline{\theta})$:

$$\upsilon\underline{\theta}\left(1-\underline{\theta}\right)h'\left(\underline{u}_A\right)=\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\overline{\theta}\left(1-\underline{\theta}\right).$$

- Subtract the latter from the former:

$$\begin{aligned}&\upsilon\underline{\theta}\left(1-\underline{\theta}\right)h'\left(\underline{u}_N\right)-\upsilon\underline{\theta}\left(1-\underline{\theta}\right)h'\left(\underline{u}_A\right)\\=&\left[\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\underline{\theta}\left(1-\overline{\theta}\right)\right]-\left[\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\overline{\theta}\left(1-\underline{\theta}\right)\right]\end{aligned}$$

  or

$$\begin{aligned}&\upsilon\underline{\theta}\left(1-\underline{\theta}\right)\left[h'\left(\underline{u}_N\right)-h'\left(\underline{u}_A\right)\right]\\=&\mu\left[\overline{\theta}\left(1-\underline{\theta}\right)-\underline{\theta}\left(1-\overline{\theta}\right)\right]=\mu\left(\overline{\theta}-\underline{\theta}\right).\end{aligned}$$

  Since $\upsilon\underline{\theta}\left(1-\underline{\theta}\right)>0$, $\overline{\theta}>\underline{\theta}$, $\mu>0$, and $h''>0$, the above inequality implies that

$$\underline{u}_N>\underline{u}_A.$$

  - That is, the low type is underinsured, which is the second one of the results we were asked to show.

**d) In some other adverse selection models that we studied, the outside option for the "good" type was (sufficiently much) more attractive than the "bad" type's outside option. This gave rise to a phenomenon called "countervailing incentives". Answer, in words, the following questions: (i) What is by meant by "countervailing incentives"? (ii) What are the possible consequences of this phenomenon in terms of efficiency and rent extraction at the second-best optimum? (iii) What is the intuition for the results under (ii)?**

- **(i)** In the standard adverse selection model, with two types who have equally attractive outside options, it is the good type that has an incentive to pass himself off as the bad type. If we instead assume that the good type has a more attractive outside option than the bad type has, then this will create an incentive in the opposite direction (a "countervailing incentive") — that model feature will add a positive value of being perceived as a good type as opposed to a bad type. If the difference in outside options is only moderate, the incentive to be perceived as a bad type is still the strongest; but for a sufficiently large difference in outside options (with the one of the good type being the more attractive), the net effect is that the agent has an incentive to pass himself off as the good type. We refer to the incentives to be perceived as the good type (created by the difference in outside options) as "countervailing incentives", regardless of whether the net effect is such that the agent wants to be perceived as a good or a bad type.

- **(ii)** As we gradually increase the extent to which the good type's outside option is more attractive than the bad type's, we obtain, in turn, the following outcomes:
  - The same outcome as in the standard model with equally attractive outside options: Efficiency for the good type but distortion downwards for the bad type; rents to the good type but no rents for the bad type.
  - Efficiency for the good type but distortion downwards for the bad type; rents to neither type.
  - Efficiency for both types; rents to neither type.
  - Efficiency for the bad type but distortion upwards for the bad type; rents to neither type.
  - The following case can arise at least if we ignore the possibility that P shuts down one of the types: Efficiency for the bad type but distortion upwards for the bad type; rents to the bad type but no rents for the bad type.
- **(iii)** One of the above results is that, for intermediate levels of difference in outside options, there is no inefficiency. The intuition for this is that the incentives to be perceived as a bad type and the countervailing incentives to be perceived as a good type are roughly equally strong and hence they cancel each other out: no type has an incentive to be perceived as another type. Therefore there are no (binding) incentive compatibility constraints, so we are effectively back in the first-best situation.
- Another one of the above results is that, for a large difference in outside options, it is the good type's quantity that is distorted (and it is distorted upwards). The intuition for this is that now the countervailing incentives are so strong that (a) it is the IC-bad constraint that binds and (b) the good type is not anymore the "money machine" — his outside option is so high that it is easier for P to earn money on the bad (and less able) type. Because of (a) P must distort at least one type's quantity and because of (b) the most profitable option for P is to distort the good type's quantity.

**Question 2 (moral hazard)**

**This is a model of so-called sharecropping. It is identical to one that we studied in the course.**

**A landlord (the principal, P) owns a piece of land and wants to lease the land to a poor farmer (the agent, A). If entering such an agreement, A can, when farming the land, choose whether to work hard ($e = 1$) and incur a cost $\psi > 0$, or not to work hard ($e = 0$) and incur no cost. Depending on whether A works hard or not and on**

**the weather, the output that is produced may be high ($q = \overline{q}$) or low ($q = \underline{q}$, with $0 \leq \underline{q} < \overline{q}$). The probability with which the output is high equals $\pi_1$ if A works hard and $\pi_0$ if A does not work hard. Assume that $0 < \pi_0 < \pi_1 < 1$. The market price of the output equals unity. Therefore, $q$ is also the market value of the output.**

**P (and the court) can observe which output that is realized ($\overline{q}$ or $\underline{q}$) but not whether A has worked hard or not. Therefore, in principle, the contract between P and A could consist of two numbers, indicating how much A should pay P in each state. However, the contract that is actually used is a so-called sharecropping contract, which is characterized by a single number, $\alpha \in [0, 1]$. The number $\alpha$ is the share of output that A is allowed to keep, whereas the remaining share $1 - \alpha$ is paid to P.**

**Therefore, P's expected profit equals**

$$V_e = (1 - \alpha) \left[\pi_e \overline{q} + (1 - \pi_e) \underline{q}\right] \qquad \text{for } e \in \{0, 1\}.$$

**Moreover, A's expected utility equals $U_1 = \alpha \left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right] - \psi$ if working hard and it equals $U_0 = \alpha \left[\pi_0 \overline{q} + (1 - \pi_0) \underline{q}\right]$ if not working hard. A's outside option would yield the utility zero. A is protected by limited liability, meaning that a contract cannot stipulate that A must pay, in net terms, some amount of money to P. It is assumed that P has all the bargaining power and makes a take-it-or-leave-it offer to A.**

a) **Explain why, given the assumed contract form and the assumption that $\alpha \in [0, 1]$, the limited liability constraint is automatically satisfied.**

- If A is protected by limited liability, then that means, in this model, that "a contract cannot stipulate that A must pay, in net terms, some amount of money to P". In other words, the net amount that is paid to A must be non-negative. However, in this model the payment to A equals a non-negative number $\alpha$ multiplied by a non-negative profit-level. Therefore, also the payment to A must be non-negative.

b) **Suppose P does not want to induce A to work hard. Formulate P's optimization problem in this situation, solve the problem, and show that P's expected profit at the optimum equals $V_0^* = \pi_0 \overline{q} + (1 - \pi_0) \underline{q}$.**

- If the landlord does not want to induce the tenant to work hard, his problem is to choose $\alpha \in [0, 1]$ so as to to maximize

$$\boxed{V_0 = (1 - \alpha) \left[\pi_0 \overline{q} + (1 - \pi_0) \underline{q}\right]}$$

subject to the tenant's individual rationality constraint:

$$U_0 = \boxed{\alpha \left[\pi_0 \overline{q} + (1 - \pi_0) \underline{q}\right] \geq 0.} \qquad \text{(IR-L)}$$

- The objective is decreasing in $\alpha$. It is therefore optimal to lower $\alpha$ until any of the constraints says stop, which happens at $\alpha = 0$ (at that value of $\alpha$, both IR-L and the constraint requiring that $\alpha \geq 0$ are binding).
- The solution is therefore to set $\alpha = 0$.
- P's expected profit at the optimum is obtained by evaluating the objective $V_0$ at $\alpha = 0$:

$$\boxed{V_0^* = \pi_0 \overline{q} + (1 - \pi_0) \underline{q}.}$$

**c) Suppose P does want to induce A to work hard. Formulate P's optimization problem in this situation, solve the problem, and show that P's expected profit at the optimum equals**

$$V_1^* = \left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right] - \frac{\left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right] \psi}{(\pi_1 - \pi_0) \left(\overline{q} - \underline{q}\right)}.$$

- If the landlord *does* want to induce the tenant to work hard, his problem is to choose $\alpha \in [0, 1]$ so as to to maximize

$$\boxed{V_1 = (1 - \alpha) \left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right]}$$

subject to the tenant's individual rationality constraint:

$$U_1 = \boxed{\alpha \left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right] - \psi \geq 0} \qquad \text{(IR-H)}$$

and his moral hazard incentive compatibility constraint:

$$\begin{aligned} U_1 &= \alpha \left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right] - \psi \\ &\geq U_0 = \alpha \left[\pi_0 \overline{q} + (1 - \pi_0) \underline{q}\right] \\ &\Leftrightarrow \boxed{\alpha (\pi_1 - \pi_0) \left(\overline{q} - \underline{q}\right) \geq \psi.} \end{aligned} \qquad \text{(IC)}$$

- Since $U_0 \geq 0$, $\boxed{\text{IC implies IR-H and hence we can ignore IR-H}}$.
- Restate the problem: Maximize

$$\boxed{V_1 = (1 - \alpha) \left[\pi_1 \overline{q} + (1 - \pi_1) \underline{q}\right]}$$

w.r.t. $\alpha$, subject to the tenant's moral hazard incentive compatibility constraint

$$\boxed{\alpha (\pi_1 - \pi_0) \left(\overline{q} - \underline{q}\right) \geq \psi.} \qquad \text{(IC)}$$

- By inspection [spell out the arguments, students!], the optimal $\alpha$ is such that IC binds:

$$\alpha (\pi_1 - \pi_0) \left(\overline{q} - \underline{q}\right) = \psi \Leftrightarrow \boxed{\alpha^{SB} = \frac{\psi}{(\pi_1 - \pi_0)\left(\overline{q} - \underline{q}\right)}.}$$

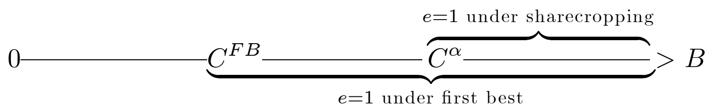
- The optimal $\alpha$ is chosen to make sure that the tenant (just barely) prefers to work hard.

- P's expected profit at the optimum is obtained by evaluating the objective $V_1$ at $\alpha = \alpha^{SB}$:

$$\boxed{V_1^* = \left[\pi_1 \overline{q} + (1-\pi_1)\,\underline{q}\right] - \frac{\left[\pi_1 \overline{q} + (1-\pi_1)\underline{q}\right]\psi}{(\pi_1-\pi_0)\left(\overline{q}-\underline{q}\right)}.}$$

**d) Explain, in words, in what sense the sharecropping contract form gives rise to underprovision of effort relative to both the second best optimum (i.e., the optimum given unobservable effort and a contract with two numbers) and relative to the first best optimum (i.e., the optimum given observable effort and a contract with two numbers). Also explain the intuition for (i.e., the logic behind) each one of those two results.**

- In each case we have "underprovision of effort" in the sense that there are some parameter values for which effort is not provided ($e = 0$) with the optimal sharecropping contract whereas effort is indeed provided ($e = 1$) with, respectively, the optimal second-best contract and the optimal first-best contract. The figure below illustrates this for the latter case (where $B$ is the benefit of implementing $e = 1$, $C^{FB}$ is the first-best cost of implementing $e = 1$, and $C^{\alpha}$ is the cost of implementing $e = 1$ when using a sharecropping contract):




So in this figure we have underprovision whenever $B \in \left(C^{FB}, C^{\alpha}\right)$.

- The reason why we have underprovision with the optimal sharecropping contract relative the optimal second-best contract is that P has access to a smaller number of instruments in the former case: P can choose only a single contract variable $\alpha$, rather than two independent contract variables. To do a given task (namely, to provide A with the proper incentives) is harder and therefore more costly to do with a single instruments than with two instruments.

- The reason why we have underprovision with the optimal sharecropping contract relative the optimal first-best contract is partly the same as the reason explained in the bullet point above. But here there is also some further underprovision, because there is underprovision with the optimal second-best contract relative the optimal first-best contract. The reason for that underprovision is that under first best (but not second best) P can observe A's chosen effort level, which means that P can simply write the required $e$ into the contract and does not need to bother about any incentive compatibility (IC) constraint. The fact that there is no IC constraint makes it easier and hence less costly to implement $e = 1$.

END OF EXAM